Assignment 5 Solutions

1. a) We have $1 = \int_{-\infty}^{\infty} p_\theta(t)dt$ So lets compute the integral.

$$1 = \int_{-\infty}^{\infty} p_\theta(t)dt$$

$$= \underbrace{\int_{-\infty}^{0} p_\theta(t)\,dt}_{0} + \int_{0}^{2\pi} p_\theta(t)dt + \underbrace{\int_{2\pi}^{\infty} p_\theta(t)dt}_{0}$$

$$= \int_0^{2\pi} c\sin^2\left(\frac{t}{2}\right)dt$$

$$= \int_0^{2\pi} c\left(\frac{1}{2} - \frac{1}{2}\cos t\right)dt \quad , \text{ since } \cos\theta = 1 - 2\sin^2\left(\frac{\theta}{2}\right)$$

$$= c\left[\frac{t}{2} - \frac{\sin t}{2}\right]_0^{2\pi}$$

$$= c\left[\left(\frac{2\pi}{2} - \frac{\sin 2\pi}{2}\right) - \left(\frac{0}{2} - \frac{\sin 0}{2}\right)\right]$$

$$= c\pi$$

So $c = \frac{1}{\pi}$.

b) $F_\theta(t) = \mathbb{P}_\theta(\theta \le t)$

$$= \int_{-\infty}^{t} p_\theta(s)ds$$

- When $t < 0$, $F_\theta(t) = \int_{-\infty}^{t} p_\theta(s)\,ds = 0$, since $p_\theta(s) = 0$ when $s < 0$.

- When $0 \le t \le 2\pi$, $F_\theta(t) = \int_0^t p_\theta(s)ds$, since $p_\theta(s) = 0$, $s < 0$.

$$= \int_0^t \frac{1}{\pi}\sin^2\left(\frac{s}{2}\right)ds$$

$$= \frac{1}{\pi}\left[\frac{s}{2} - \frac{\sin s}{2}\right]_0^t \quad , \text{ by identical computation in a)}$$

$$= \frac{1}{2\pi}(t - \sin t)$$

- When $t \geq 2\pi$, $F_\theta(t) = \mathbb{P}(\theta \leq t)$

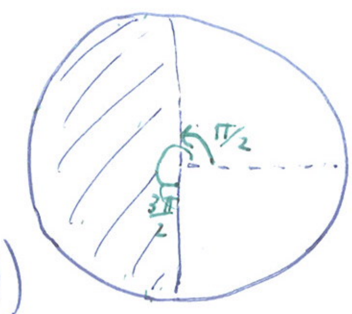
$$= \mathbb{P}(0 \leq \theta \leq 2\pi)$$

$$= 1$$

So $F_\theta(t) = \begin{cases} 0 & , \ t < 0 \\ \frac{1}{2\pi}(t - \sin t) & , \ 0 \leq t \leq 2\pi \\ 1 & , \ 2\pi \leq t \end{cases}$

c) $\mathbb{P}$( spinner lands on left hand of board)

$= \mathbb{P}\left(\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}\right)$ ,




$= F_\theta\left(\frac{3\pi}{2}\right) - F_\theta\left(\frac{\pi}{2}\right)$

$= \frac{1}{2\pi}\left(\frac{3\pi}{2} - \sin\left(\frac{3\pi}{2}\right)\right) - \frac{1}{2\pi}\left(\frac{\pi}{2} - \sin\left(\frac{\pi}{2}\right)\right)$

$= \frac{1}{2\pi}\left(\frac{3\pi}{2} - (-1)\right) - \frac{1}{2\pi}\left(\frac{\pi}{2} - 1\right)$

$= \frac{1}{2\pi}(\pi + 2)$

$= \frac{1}{2} + \frac{1}{\pi}$

$\approx 0.818\ldots$

$$\mathbb{P}(\tan\theta > 1) = \mathbb{P}\left(\pi/4 < \theta < \frac{\pi}{2} \text{ or } \frac{5\pi}{4} < \theta < \frac{3\pi}{2}\right)$$

$$= \mathbb{P}\left(\frac{\pi}{4} < \theta < \frac{\pi}{2}\right) + \mathbb{P}\left(\frac{5\pi}{4} < \theta < \frac{3\pi}{2}\right)$$

$$= F_\theta\left(\frac{\pi}{2}\right) - F_\theta\left(\frac{\pi}{4}\right) + F_\theta\left(\frac{3\pi}{2}\right) - F_\theta\left(\frac{5\pi}{4}\right)$$

$$= \frac{1}{2\pi}\left(\frac{\pi}{2} - \sin\left(\frac{\pi}{2}\right)\right) - \frac{1}{2\pi}\left(\frac{\pi}{4} - \sin\left(\frac{\pi}{4}\right)\right)$$

$$+ \frac{1}{2\pi}\left(\frac{3\pi}{2} - \sin\left(\frac{3\pi}{2}\right)\right) - \frac{1}{2\pi}\left(\frac{5\pi}{4} - \sin\left(\frac{5\pi}{4}\right)\right)$$

$$= \frac{1}{2\pi}\left(\frac{\pi}{2} - 1\right) - \frac{1}{2\pi}\left(\frac{\pi}{4} - \frac{1}{\sqrt{2}}\right)$$

$$+ \frac{1}{2\pi}\left(\frac{3\pi}{2} - (-1)\right) - \frac{1}{2\pi}\left(\frac{5\pi}{4} - \left(-\frac{1}{\sqrt{2}}\right)\right)$$

$$= \frac{1}{2\pi}\left(\frac{\pi}{2} + \frac{3\pi}{2} - \frac{\pi}{4} - \frac{5\pi}{4}\right)$$

$$= \frac{1}{2\pi}\left(\frac{\pi}{2}\right)$$

$$= \frac{1}{4}$$

d) $\mathbb{E}(\theta) = \int_{-\infty}^{\infty} t\, p_\theta(t)\,dt$

$$= \int_0^{2\pi} \frac{t}{\pi}\sin^2\left(\frac{t}{2}\right)dt \quad , \quad u = \frac{t}{\pi} \qquad dv = \sin^2\left(\frac{t}{2}\right)dt$$

$$du = \frac{dt}{\pi} \qquad v = \int \sin^2\left(\frac{t}{2}\right)dt$$

$$= \int \frac{1}{2} - \frac{1}{2}\cos t\, dt$$

$$= \frac{t}{2} - \frac{1}{2}\sin t$$

$$= \frac{t}{\pi}\left(\frac{t}{2} - \frac{1}{2}\sin t\right)\Big|_0^{2\pi} - \frac{1}{2\pi}\int_0^{2\pi} t - \sin t\, dt$$

$$= \frac{2\pi}{\pi}\left(\frac{2\pi}{2} - \frac{1}{2}\cancel{\sin 2\pi}^{\,0}\right) - \frac{1}{2\pi}\left[\frac{t^2}{2} + \cos t\right]_0^{2\pi}$$

$$\mathbb{E}(\theta) = \frac{(2\pi)^2}{2\pi} - \frac{1}{2\pi}\left[\left(\frac{(2\pi)^2}{2} + \cos 2\pi\right) - \left(\frac{0^2}{2} + \cos 0\right)\right]$$

$$= 2\pi - \underbrace{\frac{1}{2\pi}\left(\frac{(2\pi)^2}{2}\right.}_{\pi} \left. + \underbrace{1 - 1}_{0}\right)$$

$$= 2\pi - \pi$$

$$= \pi$$

Which is not surprising since $p_\theta(t)$ is symetric about $\pi$.

e) Bonus:

If I play the game enough times I want my average winnings to be non-negative. Let $Y$ denote my winnings after on spin.

So $Y = 100\cos^2(\frac{\theta}{2})$

So I want the averge winnings of $Y$ to be greater than the amount I paid. So I want

$$\mathbb{E}(Y) \geq k.$$

Thus largest $k$ I should be willing to pay is $\mathbb{E}(Y)$. Let find $\mathbb{E}(Y)$.

$$\mathbb{E}(Y) = \mathbb{E}\left(100\cos^2\left(\tfrac{\theta}{2}\right)\right)$$

$$= \int_{-\infty}^{\infty} 100\cos^2\left(\tfrac{t}{2}\right) p_\theta(t)\,dt$$ , by definition of $\mathbb{E}(f(X))$ (in this case $f(x) = 100\cos^2(\frac{x}{2})$).

$$= \int_0^{2\pi} 100\cos^2\left(\tfrac{t}{2}\right)\frac{1}{\pi}\sin^2\left(\tfrac{t}{2}\right)dt$$

$$= \frac{100}{\pi}\int_0^{2\pi} \cos^2\left(\frac{t}{2}\right)\sin^2\left(\frac{t}{2}\right)dt$$

$$= \frac{100}{\pi}\int_0^{2\pi}\left(\cos\left(\frac{t}{2}\right)\sin\left(\frac{t}{2}\right)\right)^2 dt$$

$$= \frac{100}{\pi}\int_0^{2\pi}\left(\frac{\sin t}{2}\right)^2 dt \quad \text{, since } 2\sin\theta\cos\theta = \sin 2\theta$$

$$= \frac{25}{\pi}\int_0^{2\pi}\sin^2 t\, dt$$

$$= \frac{25}{\pi}\int_0^{2\pi}\frac{1}{2} - \frac{1}{2}\cos(2t)\, dt$$

$$= \frac{25}{\pi}\left[\frac{t}{2} - \frac{1}{4}\sin(2t)\right]_0^{2\pi}$$

$$= \frac{25}{\pi}\left[\left(\frac{2\pi}{2} - \overset{0}{\cancel{\frac{1}{4}\sin(2(2\pi))}}\right) - \left(\frac{0}{2} - \overset{0}{\cancel{\frac{1}{4}\sin 0}}\right)\right]$$

$$= \frac{25\cdot 2\pi}{\pi\cdot 2}$$

$$= 25$$

So you should pay maximum 25 dollars.

2. Let $\mu = \mathbb{E}(X)$. So

$$\mathrm{Var}(X) = \mathbb{E}\left((X-\mu)^2\right)$$

$$= \int_{-\infty}^{\infty} (t-\mu)^2 \rho_X(t)\,dt$$

$$= \int_{-\infty}^{\infty} (t^2 - 2\mu t + \mu^2)\, \rho_X(t)\,dt$$

$$= \underbrace{\int_{-\infty}^{\infty} t^2 \rho_X(t)\,dt}_{\mathbb{E}(X^2)} - 2\mu \underbrace{\int_{-\infty}^{\infty} t\, \rho_X(t)\,dt}_{\mathbb{E}(X)} + \mu^2 \underbrace{\int_{-\infty}^{\infty} \rho_X(t)\,dt}_{1}$$

$$= \mathbb{E}(X^2) - 2\mu\, \mathbb{E}(X) + \mu^2$$

$$= \mathbb{E}(X^2) - 2\,\mathbb{E}(X)^2 + \mathbb{E}(X)^2, \text{ since } \mu = \mathbb{E}(X)$$

$$= \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

3. a) • $p_X(t) = \begin{cases} \lambda e^{-\lambda t} & , t \geq 0 \\ 0 & , t < 0 \end{cases}$

When $t < 0$, $F_X(t) = \int_{-\infty}^{t} p_X(s)ds$

$= 0$ , since $p_X(s) = 0$, $s < 0$

When $t \geq 0$, $F_X(t) = \int_{-\infty}^{t} p_X(s)ds$

$= \int_0^t \lambda e^{-\lambda s} ds$

$= \left[ -e^{-\lambda s} \right]_0^t$

$= 1 - e^{-\lambda t}$

So $F_X(t) = \begin{cases} 0 & , t < 0 \\ 1-e^{-\lambda t} & , t \geq 0 \end{cases}$

• $\mathbb{E}(X) = \int_{-\infty}^{\infty} t \, p_X(t) dt$

$= \int_0^{\infty} t \lambda e^{-\lambda t} dt$ , $u = t$ , $dv = \lambda e^{-\lambda t} dt$

$du = dt$ $\quad v = -e^{-\lambda t}$

$= -te^{-\lambda t} \Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda t} dt$

$= -\frac{t}{e^{\lambda t}} \Big|_0^{\infty} + \left[ -\frac{e^{-\lambda t}}{\lambda} \right]_0^{\infty}$

$= \lim_{b \to \infty} \left[ \frac{-t}{e^{\lambda t}} - \frac{e^{-\lambda t}}{\lambda} \right]_0^b$

$$= \lim_{b\to\infty} \left[\frac{-b}{e^{\lambda b}} - \frac{e^{-\lambda b}}{\lambda}\right] - \left[\frac{-0}{e^0} - \frac{e^0}{\lambda}\right]$$

$$= \lim_{b\to\infty} -\frac{b}{e^{\lambda b}} - \frac{e^{-\lambda b}}{\lambda} + \frac{1}{\lambda}$$

$$= \lim_{b\to\infty} \frac{-b}{e^{\lambda b}} + \frac{1}{\lambda} \quad \text{, since } e^{-\lambda b} \to 0 \text{ as } b \to \infty$$

$$= \lim_{b\to\infty} \frac{-1}{\lambda e^{\lambda b}} + \frac{1}{\lambda} \quad \text{, by l'hopital's rule}$$

$$= \frac{1}{\lambda}$$

- $$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} t^2 p_X(t)\,dt$$

$$= \int_0^{\infty} t^2 \lambda e^{-\lambda t}\,dt$$

$$= \lim_{b\to\infty} \int_0^b t^2 \lambda e^{-\lambda t}\,dt \quad , \quad \begin{array}{ll} u = t^2, & dv = \lambda e^{-\lambda t}dt \\ du = 2t\,dt & v = -e^{-\lambda t} \end{array}$$

$$= \lim_{b\to\infty} -t^2 e^{-\lambda t}\Big|_0^b + \int_0^b 2te^{-\lambda t}\,dt$$

$$= \lim_{b\to\infty} \frac{-b^2}{e^{\lambda b}} + \lim_{b\to\infty} \frac{2}{\lambda}\int_0^b t\lambda e^{-\lambda t}\,dt$$

$$= \lim_{b\to\infty} \frac{-b^2}{e^{\lambda b}} + \frac{2}{\lambda}\underbrace{\int_0^{\infty} t\lambda e^{-\lambda t}\,dt}_{\mathbb{E}(X) = \frac{1}{\lambda}}$$

$$= \lim_{b\to\infty} \frac{-b^2}{e^{\lambda b}} + \frac{2}{\lambda^2}$$

$$= \lim_{b\to\infty} \frac{-2b}{\lambda e^{\lambda b}} + \frac{2}{\lambda^2} \quad \text{, by l'hopital}$$

$$= \lim_{b\to\infty} \frac{-2}{\lambda^2 e^{\lambda b}} + \frac{2}{\lambda^2} \quad , \text{ by l'hopital}$$

$$= 0 + \frac{2}{\lambda^2}$$

Thus by question 2.

$$Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$
$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2$$
$$= \frac{1}{\lambda^2}$$

Note you don't need to know l'hopital for the final.

b) • $F_Y(t) = \begin{cases} 0 & , t < 0 \\ t^4 & , 0 \le t \le 1 \\ 1 & , t \ge 1 \end{cases}$

$$p_Y(t) = F_Y'(t)$$
$$= \begin{cases} 0 & , t < 0 \\ 4t^3 & , 0 \le t \le 1 \\ 0 & , 1 \le t \end{cases}$$

• $E(Y) = \int_{-\infty}^{\infty} t \, p_Y(t) \, dt$

$$= \int_0^1 t \, 4t^3 \, dt$$
$$= 4 \int_0^1 t^4 \, dt$$
$$= 4 \left[ \frac{t^5}{5} \right]_0^1$$
$$= \frac{4}{5}$$

$$\mathbb{E}(Y^2) = \int_{-\infty}^{\infty} t^2 \, p_Y(t)\,dt$$

$$= \int_0^1 t^2 \, 4t^3 \, dt$$

$$= 4 \int_0^1 t^5 \, dt$$

$$= 4 \left[ \frac{t^6}{6} \right]_0^1$$

$$= \frac{4}{6}$$

$$= \frac{2}{3}$$

So

$$\mathrm{Var}(Y) = \mathbb{E}(Y^2) - \mathbb{E}(Y)^2$$

$$= \frac{2}{3} - \left(\frac{4}{5}\right)^2$$

$$= \frac{2}{3} - \frac{16}{25}$$

$$= \frac{2}{75}$$

4. a) $\mathbb{E}(Y) = \mathbb{E}(aX+b)$

$$= \int_{-\infty}^{\infty} (at+b)\, p_X(t)\, dt$$

$$= a \underbrace{\int_{-\infty}^{\infty} t\, p_X(t)\, dt}_{\mathbb{E}(X)} + b \underbrace{\int_{-\infty}^{\infty} p_X(t)\, dt}_{1}$$

$$= a\,\mathbb{E}(X) + b$$

b) $\mathrm{Var}(Y) = \mathbb{E}[(Y - \mathbb{E}(Y))^2]$

$$= \mathbb{E}[(aX+b - \mathbb{E}(aX+b))^2]$$

$$= \mathbb{E}[(aX+b - a\mathbb{E}(X) - b)^2]$$

$$= \mathbb{E}[(aX - a\mathbb{E}(X))^2]$$

$$= \mathbb{E}[a^2(X - \mathbb{E}(X))^2]$$

$$= a^2\,\mathbb{E}[(X - \mathbb{E}(X))^2] \quad , \text{ by a)}$$

$$= a^2\,\mathrm{Var}(X)$$

c) i) $Y$ is the amount you spend on the burger (\$3.87)

plus the value of your time waiting for the burger (\$(0.25)15)

plus the value of your time waiting in line. (\$ $0.25X$).

So $Y = 3.87 + (0.25)15 + 0.25X$

$$= 7.62 + 0.25X$$

ii) $\mathbb{E}(Y) = \mathbb{E}(7.62 + 0.25X)$

$= 7.62 + 0.25\,\mathbb{E}(X)$ , by a)

Since $X$ has exponential distribution with $\lambda = \frac{1}{30}$ , by 3a)

$$\mathbb{E}(X) = \frac{1}{\lambda} = \frac{1}{\frac{1}{30}} = 30$$

$$Var(X) = \frac{1}{\lambda^2} = \frac{1}{\left(\frac{1}{30}\right)^2} = 30^2$$

So $\mathbb{E}(Y) = 7.62 + 0.25(30)$

$= \$15.12$

iii) $\sigma(Y) = \sqrt{Var(Y)}$

$= \sqrt{Var(7.62 + 0.25X)}$

$= \sqrt{(0.25)^2\,Var(X)}$ , by b)

$= \sqrt{(0.25)^2(30)^2}$

$= (0.25)(30)$

$= \$7.50$